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SOLUTION SKETCHES TO HOMEWORK 1

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Homework 1.1 (An example of propagation of convergence*)**.** Let $D \subset \mathbf{C}$ be a bounded domain and denote by $\bar{D}$ is closure. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of continuous functions $f_n : \bar{D} \to \mathbf{C}$ such that f_n is holomorphic in D for every $n \in \mathbf{N}$. Assume $(f_n)_{n\in\mathbf{N}}$ converges uniformly on the boundary ∂D.

a. Show $(f_n)_{n\in\mathbf{N}}$ converges uniformly on $\bar{D}$ to some function $f : \bar{D} \to \mathbf{C}$[1].
b. Show f is holomorphic on D.

Homework 1.2 (Sequences of holomorphic functions)**.**

a. Let $(f_n)_{n\in\mathbf{N}}$ constitute a sequence of holomorphic functions $f_n : B_1(0) \to \mathbf{C}$ converging locally uniformly to a given holomorphic function $f : B_1(0) \to \mathbf{C}$. Does the sequence $(f_n^{(n)})_{n\in\mathbf{N}}$ of derivatives of f_n with increasing order converge locally uniformly to a continuous function $g : B_1(0) \to \mathbf{C}$? Give a proof or find a counterexample.
b. Give an example of an open set $U \subset \mathbf{C}$ and a sequence $(f_n)_{n\in\mathbf{N}}$ of holomorphic functions $f_n : U \to \mathbf{C}$ that converges locally uniformly to a function $f : U \to \mathbf{C}$ and such that f_n has exactly one zero for every $n \in \mathbf{N}$ while f has no zero. Can one construct a counterexample under the requirement of uniform convergence on U?

Solution. a. *False.* Consider the sequence defined by $f_n(z) := z^n$, which converges locally uniformly to 0 on $B_1(0)$. However, the n-th derivative of f_n computes as $f_n^{(n)}(z) = n!$, which is not even bounded as $n \to \infty$.

b. We take $U := \mathbf{C}$ and set $f_n(z) := 1 + z/n$. Then f_n has its only zero at $-n$ for every $n \in \mathbf{N}$, but the sequence converges locally uniformly to the function constantly equal to one on U, which has no zero.

To have an example certifying uniform convergence, one can take $f_n(z) := z^2 - 1/n$ on the open halfspace $U := \{\Re > 0\}$.

Homework 1.3 (Convergence of varying path-integrals)**.** Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of holomorphic functions $f_n : U \to \mathbf{C}$ that converges locally uniformly on a given open set $U \subset \mathbf{C}$ to some function $f : U \to \mathbf{C}$. Moreover, let $(\gamma_n)_{n\in\mathbf{N}}$ be a sequence of C^1-paths $\gamma_n : [0,1] \to U$ such that $\gamma_n \to \gamma$ and $\gamma_n' \to \gamma'$ uniformly on $[0,1]$ as $n \to \infty$, where $\gamma : [0,1] \to U$ is a C^1-path. Show

$$\lim_{n\to\infty} \int_{\gamma_n} f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Solution. By definition of the complex path integral, every $n \in \mathbf{N}$ satisfies

$$\int_{\gamma_n} f_n(z)\,\mathrm{d}z = \int_0^1 f_n \circ \gamma_n(t)\,\gamma_n'(t)\,\mathrm{d}t.$$

The idea of the proof will be to apply Lebesgue's dominated convergence theorem to the right-hand side. To this aim, we estimate the factors $f_n \circ \gamma_n$ and γ_n' separately.

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[1]**Hint.** Show $(f_n)_{n\in\mathbf{N}}$ is a Cauchy sequence with respect to uniform convergence.

The latter is elementary. Since $\gamma_n' \to \gamma'$ uniformly on $[0,1]$ as $n \to \infty$, the sequence $(|\gamma_n'(t)|)_{n\in\mathbf{N}}$ is bounded uniformly in $t \in [0,1]$; in other words,

$$\sup_{n\in\mathbf{N}} \sup_{t\in[0,1]} |\gamma_n'(t)| < \infty.$$

We turn to the term $f_n \circ \gamma_n$. Here, as $(f_n)_{n\in\mathbf{N}}$ converges only *locally* uniformly, some more work is needed. The idea is that for all sufficiently large $n \in \mathbf{N}$, all segments $\gamma_n([0,1])$ will be contained in a neighborhood of the segment $\gamma([0,1])$ which is compactly contained in U; on this neighborhood, $(f_n)_{n\in\mathbf{N}}$ converges uniformly. To construct this neighborhood, observe first that γ is continuous (as a uniform limit of continuous paths). Hence, since $[0,1]$ is compact, so is the image $\gamma([0,1])$.

We claim there exists $\delta > 0$ such that the closure $\bar{B}_\delta(\gamma([0,1]))$ of the δ-neighborhood of $\gamma([0,1])$ is contained in U. This is clear if $U = \mathbf{C}$; hence, let us assume $U \neq \mathbf{C}$. Let $\mathsf{d}_{\mathbf{C}\setminus U}\colon \mathbf{C} \to [0,\infty)$ denote the distance function to the complement of U, defined by

$$\mathsf{d}_{\mathbf{C}\setminus U}(z) := \inf_{y\in\mathbf{C}\setminus U} |z-y|.$$

Then $\mathsf{d}_{\mathbf{C}\setminus U}$ is Lipschitz continuous[2]. Indeed, let $z, z' \in \mathbf{C}$. Since $\mathsf{d}_{\mathbf{C}\setminus U}(z')$ is finite, given any $\varepsilon > 0$ there exists an ε-almost minimizer $y_\varepsilon \in \mathbf{C}\setminus U$ in the definition of $\mathsf{d}_{\mathbf{C}\setminus U}$, i.e.

$$|z' - y_\varepsilon| \leq \mathsf{d}_{\mathbf{C}\setminus U}(z') + \varepsilon.$$

This easily implies

$$\mathsf{d}_{\mathbf{C}\setminus U}(z) - \mathsf{d}_{\mathbf{C}\setminus U}(z') \leq |z - y_\varepsilon| - |z' - y_\varepsilon| + \varepsilon \leq |z - z'| + \varepsilon;$$

here, we used the triangle inequality in the second estimate. The arbitrariness of ε gives

$$\mathsf{d}_{\mathbf{C}\setminus U}(z) - \mathsf{d}_{\mathbf{C}\setminus U}(z') \leq |z - z'|.$$

Flipping the roles of z and z' in the previous argument gives

$$|\mathsf{d}_{\mathbf{C}\setminus U}(z) - \mathsf{d}_{\mathbf{C}\setminus U'}(z')| \leq |z - z'|,$$

which is the desired Lipschitz continuity. In turn, $\mathsf{d}_{\mathbf{C}\setminus U}$ is continuous on the compact set $\gamma([0,1])$. Since U is open and $\gamma([0,1])$ is contained in U, $\mathsf{d}_{\mathbf{C}\setminus U}$ is positive on $\gamma([0,1])$ and thus, since continuous functions attain their minima on compact sets, uniformly bounded from zero by 2δ, where $\delta > 0$. Considering the δ- instead of the 2δ-neighborhood in order to ensure the closure $\bar{B}_\delta(\gamma([0,1]))$ is still contained in U yields the claim.

Now by uniform convergence of γ_n to γ, there exists $n_\delta \in \mathbf{N}$ such that for every $n' \geq n_\delta$,

$$\sup_{t\in[0,1]} |\gamma_n(t) - \gamma(t)| \leq \delta;$$

in other words, $\gamma_n([0,1]) \subset \bar{B}_\delta(\gamma([0,1]))$. Since $(f_n)_{n\in\mathbf{N}}$ converges uniformly to f on $\bar{B}_\delta(\gamma([0,1]))$ as remarked in the lecture, it is bounded on this set; therefore

$$\sup_{n\geq n_\delta} \sup_{t\in[0,1]} |f_n \circ \gamma_n(t)| \leq \sup_{n\geq n_\delta} \sup_{z\in\bar{B}_\delta(\gamma([0,1]))} |f_n(z)| < \infty.$$

Lebesgue's dominated convergence theorem thus yields

$$\begin{aligned}
\lim_{n\to\infty} \int_{\gamma_n} f_n(z)\,\mathrm{d}z &= \lim_{n\to\infty} \int_0^1 f_n \circ \gamma_n(t)\, \gamma_n'(t)\,\mathrm{d}t \\
&= \int_0^1 f \circ \gamma(t)\, \gamma'(t)\,\mathrm{d}t \\
&= \int_\gamma f(z)\,\mathrm{d}z
\end{aligned}$$

provided $f_n \circ \gamma_n \to f \circ \gamma$ and $\gamma_n \to \gamma$ pointwise on $[0,1]$ as $n \to \infty$.

[2]We will only need continuity, but the proof is a good exercise in dealing with almost minimizers or almost maximizers.

The latter holds trivially by uniform convergence.

To show the former, let $\varepsilon > 0$. Note that f is continuous on U as the locally uniform limit of continuous functions, hence uniformly continuous on the compact set $\bar{B}_\delta(\gamma([0,1]))$. Up to shrinking δ, we may and will assume every $t \in [0,1]$ and every $z \in \bar{B}_\delta(\gamma(t))$ satisfy

$$|f \circ \gamma(t) - f(z)| \leq \varepsilon.$$

In turn, by if $n \geq n_\delta$ we obtain

$$\begin{aligned} |f_n \circ \gamma_n(t) - f \circ \gamma(t)| &\leq |f_n \circ \gamma_n(t) - f \circ \gamma_n(t)| + |f \circ \gamma_n(t) - f \circ \gamma(t)| \\ &\leq \sup_{z \in \bar{B}_\delta(\gamma([0,1]))} |f_n(z) - f(z)| + \varepsilon. \end{aligned}$$

As $n \to \infty$, the right-hand side converges to ε. Since ε is arbitrary, the desired pointwise[3] convergence follows.

Homework 1.4 (Osgood's theorem)**.** Let $U \subset \mathbf{C}$ be open and $(f_n)_{n\in\mathbf{N}}$ be a sequence of holomorphic functions $f_n\colon U \to \mathbf{C}$ that converges pointwise to $f\colon U \to \mathbf{C}$. The goal of this exercise is to show there exists an open, dense subset $U_0 \subset U$ such that $(f_n)_{n\in\mathbf{N}}$ is locally uniformly bounded on U_0. As we will see in the course, this implies the local uniform convergence of $(f_n)_{n\in\mathbf{N}}$ to f on U_0; in particular, f is holomorphic on U_0.

a. Define the set of points where $(f_n)_{n\in\mathbf{N}}$ is locally uniformly bounded, i.e.,

$$U_0 := \{z \in U : \text{there exists } r > 0 \text{ with } \sup_{n\in\mathbf{N}} \sup_{z' \in B_r(z)} |f_n(z')| < \infty\}.$$

Show U_0 is open.

b. Show if U_0 is not dense in U, then there exists a ball $B_{r_0}(z_0) \subset U$ such that for all balls $B_{r'}(z') \subset B_{r_0}(z_0)$ the sequence $(f_n)_{n\in\mathbf{N}}$ is not uniformly bounded on $B_{r'}(z')$.

c. Use b. to construct a sequence of nested closed balls $\bar{B}_{r_k}(z_k) \subset B_{r_{k-1}}(z_{k-1})$ and a subsequence $(f_{n_k})_{k\in\mathbf{N}}$ such that $|f_{n_k}| \geq k$ on $\bar{B}_{r_k}(z_k)$.

d. Show the intersection of the sequence of closed balls from c. is nonempty. Derive a contradiction to the pointwise convergence of $(f_n)_{n\in\mathbf{N}}$ and conclude the proof.

Solution. a. Let $z \in U_0$ and $r > 0$ be as in the definition of U_0. We claim $B_r(z) \subset U_0$. Indeed, given any $y \in B_r(z)$ there is $r_y > 0$ such that $B_{r_y}(y) \subset B_r(z)$. Consequently,

$$\sup_{n\in\mathbf{N}} \sup_{z' \in B_{r_y}(y)} |f_n(z')| \leq \sup_{n\in\mathbf{N}} \sup_{z' \in B_r(z)} |f_n(z')| < \infty,$$

The definition of U_0 entails $y \in U_0$. Since y was arbitrary, this shows $B_r(z) \subset U_0$; since z was arbitrary this shows the claimed openness of U_0.

b. Since U_0 is not dense in U, there exist $z_0 \in U$ and $r_0 > 0$ such that $B_{r_0}(z_0) \cap U_0 = \emptyset$. Now suppose to the contrary that $(f_n)_{n\in\mathbf{N}}$ is uniformly bounded on a ball $B_{r'}(z')$ as stated. Then by definition of U_0, we would have $z' \in B_{r_0}(z_0) \cap U_0$, a contradiction.

c. Since the sequence $(f_n)_{n\in\mathbf{N}}$ is not uniformly bounded on $B_{r_0}(z_0)$, there exists $n_1 \in \mathbf{N}$ and $z_1 \in B_{r_0}(z_0)$ with the property $|f_{n_1}(z_1)| > 1$. As the function f_{n_1} is continuous, we find a closed ball $\bar{B}_{r_1}(z_1) \subset B_{r_0}(z_0)$ such that $|f_{n_1}| > 1$ on $\bar{B}_{r_1}(z_1)$. By b., the sequence $(f_n)_{n\in\mathbf{N}}$ is not uniformly bounded on the interior $B_{r_1}(z_1)$; thus, there exist $n_2 \in \mathbf{N}$ and $z_2 \in B_{r_1}(z_1)$ such that $|f_{n_2}(z_2)| > 2$. By continuity of f_{n_2} we find a closed ball $\bar{B}_{r_2}(z_2) \subset B_{r_1}(z_1)$ such that $|f_{n_2}| > 2$ on $\bar{B}_{r_2}(z_2)$. The two desired (sub)sequences are constructed by iterating this procedure.

(iv) Since the sets $\bar{B}_{r_k}(z_k)$ are nested and compact, their intersection contains an element $\bar{z}$. For this element $\bar{z}$, by construction we have $|f_{n_k}(\bar{z})| > k$ for every $k \in \mathbf{N}$. This shows the sequence $(f_n(\bar{z}))_{n\in\mathbf{N}}$ cannot converge as $n \to +\infty$ (as it has an unbounded subsequence).

[3] In fact, we have just shown $f_n \circ \gamma_n \to f \circ \gamma$ uniformly on $[0,1]$.

Hence, U_0 has to be dense in U and by definition, $(f_n)_{n\in\mathbf{N}}$ is locally uniformly bounded on the set U_0 constructed above[4].

[4]Note that we did not explicitly use the holomorphy of each element of $(f_n)_{n\in\mathbf{N}}$ in the proof. However, the latter is crucial to pass from local uniform boundedness to local uniform convergence via Montel's theorem.